# TEMPERATURE FIELD IN MULTILAYER SYSTEMS WITH VARIABLE THERMOPHYSICAL PROPERTIES

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The author proposes a method for solving nonlinear heat conduction problems in which the space-time domain is divided into a series of calculation intervals with respect to time and the coordinate.

In this paper the approximate method of solving nonlinear problems proposed in [1] is further developed.

We will consider the solution of the heat conduction equation for a plate or a multilayer system composed of plates in the case when the thermophysical properties depend on temperature. Heat exchange with the surrounding medium is governed by Newton's law (boundary condition of the third kind), the ambient temperature and the heat transfer coefficient may vary in time according to an arbitrary law. The relations for the thermal conductivities λ and specific heats c are given for an $l$-layer system in the form

$$\begin{aligned} \lambda &= \lambda_1(t), \quad c = c_1(t), \quad [0;\ L_1], \\ \lambda &= \lambda_2(t), \quad c = c_2(t), \quad [L_1;\ L_2], \\ &\dots\dots\dots\dots\dots \\ \lambda &= \lambda_l(t), \quad c = c_l(t), \quad [L_{l-1};\ L_l]. \end{aligned} \tag{1}$$

Since the temperature is an unknown function of the coordinate and time, λ and c are certain unknown complex functions of x and τ.

The space-time domain is divided into m time intervals Δτ and n layers Δx; on Δx during the interval Δτ the thermophysical characteristics take constant values. After this transformation the solution of the initial nonlinear problem can be replaced by successive solutions of linear problems with the same boundary conditions as in the initial problem.

In solving these linear problems the thermophysical parameters can be chosen as follows. During the interval Δτ the parameters λ and c can be represented by functions of the coordinate only:

$$\begin{aligned} \lambda &= \lambda[t_i(x)], \quad c = c[t_i(x)], \\ \tau &= \tau_j + \Theta(\tau_{j+1} - \tau_j), \quad 0 < \Theta < 1. \end{aligned} \tag{2}$$

Functional relation (2) is determined, firstly, by the given temperature dependence of the parameters (1) and, secondly, by the temperature distribution with respect to the coordinate

$$t_i(x)\,|_{\tau=\tau_j+\Theta(\tau_{j+1}-\tau_j);\ 0<\Theta<1}.$$

The values of λ and c in the layers are given by

$$\begin{aligned} &\lambda_1, \quad c_1, \quad [x_1;\ x_2], \quad x_1 = 0, \\ &\lambda_2, \quad c_2, \quad [x_2;\ x_3], \\ \lambda = &\lambda_3, \quad c = c_3, \quad [x_3;\ x_4], \\ &\dots \quad \dots \quad \dots \\ &\lambda_{n-1}, \quad c_{n-1}, \quad [x_{n-1};\ x_n], \\ &\lambda_n, \quad c_n, \quad [x_n;\ x_{n+1}], \end{aligned} \tag{2a}$$

where

$$\left.\begin{aligned} \lambda_i &= \lambda[t(x)] \\ c_i &= c[t(x)] \end{aligned}\right|_{x=x_i+\Theta(x_{i+1}-x_i);\ 0<\Theta<1}. \tag{2b}$$

Thus, functions with discontinuities of the first kind (2a), (2b) characterize the distribution of the thermal characteristics over the layers. On transition from the j-th to the (j + 1)-th time interval and from the i-th to the (i + 1)-th layer the parameters change discretely. Matching at the boundaries of the layers is achieved by introducing the condition of equal temperatures and heat fluxes at the boundaries. The temperature field at the end of the j-th time interval is the initial condition for the (j + 1)-th interval. As shown in [1], the point of the space-time domain, with respect to which the thermophysical parameters are found, may be selected either at the beginning of the time interval $\tau_j$ or at its end $\tau_{j+1}$, at the left $x_i$ or right boundary of the layer $x_{i+1}$, or at some intermediate value

$$\begin{aligned} \tau_j &< \tau < \tau_{j+1}, \\ x_i &< x < x_{i+1}. \end{aligned}$$

The arbitrary choice of this point determines the approximate nature of the method of solution for finite values of Δτ and Δx. From physical considerations it is obvious that as Δτ and Δx decrease, the expressions for the temperature fields in the linear problems will approach the solution of the initial nonlinear problem.

In solving this problem the initial condition may be taken as the zero condition; this simplification is based on the proposition that the effect of the initial temperature distribution grows weaker in the course of the process.

The temperature field in the first time interval is described by the expression obtained in [2]. In the second time interval the heat conduction equation and the boundary conditions for the symmetrical problem have the form

$$\begin{aligned} \frac{\partial t_i}{\partial \tau} &= a_{i,2}\frac{\partial^2 t_i}{\partial x^2}; \\ \tau_1 < \tau < \tau_2;& \quad x_i < x < x_{i+1}, \\ i &= 1,\ 2,\ 3,\ \dots,\ n; \end{aligned} \tag{3}$$

$$\frac{\partial t(x,\ \tau)}{\partial x} + \frac{\alpha_2}{\lambda_{12}}[t_c(\tau) - t(x,\ \tau)]_{x=0} = 0; \tag{3a}$$

$$\left.\frac{\partial t(x,\ \tau)}{\partial x}\right|_{x=x_{n+1}} = 0; \tag{3b}$$

$$t_i(x,\ \tau)|_{\tau=\tau_1} = \sum_{k=0}^{\infty} t_c^{(k)}(\tau_1) Q_{k,\,i,\,1}(x) + \sum_{k=0}^{\infty}(-1)^k t_c^{(k)}(0) \times \sum_{p=1}^{\infty} \frac{\Phi_i(\mu_{p,\,1},\ x)}{\mu_{p,\,1}^{2k+1}\varphi'(\mu_{p,\,1})}\exp(-\mu_{p,\,1}^2\tau_1); \tag{3c}$$

$$\lambda = \lambda_{i,\,2};\quad a = a_{i,\,2}; \tag{3d}$$

$$t_i(x_{i+1},\ \tau) = t_{i+1}(x_{i+1},\ \tau); \tag{3e}$$

$$\lambda_{i,\,2}\frac{\partial t_i(x_{i+1},\ \tau)}{\partial x} = \lambda_{i+1,2}\frac{\partial t_{i+1}(x_{i+1},\ \tau)}{\partial x}. \tag{3f}$$

As follows from [2], the functions $Q_{k,\,i}(x)$ are polynomials of degree 2k defined on the interval $[x_i;\ x_{i+1}]$. If x is represented in the form of a relative coordinate

$$N_i = (x_{i+1} - x)/(x_{i+1} - x_i),$$

the coefficients in $Q_{k,\,i}(x)$ will consist only of complexes of the type

$$R_i = \Delta x_i/\lambda_i;\quad M_i = (\Delta x_i)^2/a_i.$$

It is more convenient to find the solution of the problem for a multilayer system with a complex initial temperature distribution by the method of separation of variables [3]. First, we find the solution for the case when the temperature of the medium is given by a function in the form of the common term of a Maclaurin series, the boundary conditions (3a) and (3b) being replaced, respectively, by

$$\left.\frac{\partial t(x,\ \tau)}{\partial x} + \frac{\alpha_2}{\lambda_{12}}\left[\frac{t_c^{(k)}(0)}{k!}\tau^k - t(x,\ \tau)\right]\right|_{x_1=0} = 0; \tag{4}$$

$$t_i(x,\ \tau)|_{\tau=\tau_1} = t_c^{(k)}(0)\left\{\left[\frac{1}{k!}Q_{0,\,i,\,1}(x)\tau_1^k + \ldots + Q_{k-1,\,i,\,1}(x)\tau_1 + Q_{k,\,i,\,1}(x)\right] + \sum_{p=1}^{\infty}(-1)^k \times \frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,\,1}^{2k+1}\varphi'(\mu_{p,\,1})}\exp(-\mu_{p,\,1}^2\tau_1)\right\}. \tag{4a}$$

Making the change of variable

$$t_i(x,\ \tau) = \frac{t_c^{(k)}(0)}{k!}\tau^k + U_i(x,\ \tau), \tag{5}$$

we obtain an inhomogeneous differential heat condition equation for the multilayer system with homogeneous boundary conditions and initial condition (4a). Using substitution (5), we can write its solution in the form

$$t_i(x,\ \tau) = t_c^{(k)}(0)\left\{\frac{Q_{0,i,2}}{k!}\tau^k + \frac{Q_{1,i,2}}{(k-1)!}\tau^{k-1} + \ldots + \frac{Q_{k-1,i,2}}{1!}\tau + Q_{k,i,2} - \sum_{p=1}^{\infty}\frac{\tau_1^{k-1}}{(k-1)!}\left[\frac{\Phi_i(\mu_{p,2},\ x)}{\mu_{p,2}^3\varphi'(\mu_{p,2})} - \frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^3\varphi'(\mu_{p,1})}\right]\exp[-\mu_{p,2}^2(\tau-\tau_1)] + \sum_{p=1}^{\infty}\frac{\tau_1^{k-2}}{(k-2)!}\left[\frac{\Phi_i(\mu_{p,2},\ x)}{\mu_{p,2}^5\varphi'(\mu_{p,2})} - \frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^5\varphi'(\mu_{p,1})}\right] \times \exp[-\mu_{p,2}^2(\tau-\tau_1)] + \ldots + \sum_{p=1}^{\infty}(-1)^k\left[\frac{\Phi_i(\mu_{p,2},\ x)}{\mu_{p,2}^{2k+1}\varphi'(\mu_{p,2})} - \frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^{2k+1}\varphi'(\mu_{p,1})}\right]\exp[-\mu_{p,2}^2(\tau-\tau_1)] + \sum_{p=1}^{\infty}(-1)^k\frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^{2k+1}\varphi'(\mu_{p,1})} \times \exp[-\mu_{p,2}^2(\tau-\tau_1) - \mu_{p,1}^2\tau_1]\right\}. \tag{6}$$

We obtain the general solution in the second time interval by superposition of the temperature fields

$$t_i(x,\ \tau) = \sum_{k=0}^{\infty} t_c^{(k)}(\tau) Q_{k,i,2}(x) + \sum_{l=1}^{\infty}(-1)^l\sum_{k=l}^{\infty}\frac{t_c^{(k)}(0)}{(k-l)!}\tau_1^{k-l}\sum_{p=1}^{\infty}\left[\frac{\Phi_i(\mu_{p,2},\ x)}{\mu_{p,2}^{2l+1}\varphi'(\mu_{p,2})} - \frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^{2l+1}\varphi'(\mu_{p,1})}\right]\exp[-\mu_{p,2}^2(\tau-\tau_1)] + \sum_{k=0}^{\infty}(-1)^k t_c^{(k)}(0)\sum_{p=1}^{\infty}\frac{\Phi_i(\mu_{p,1},\ x)}{\mu_{p,1}^{2k+1}\varphi'(\mu_{p,1})} \times \exp[-\mu_{p,2}^2(\tau-\tau_1) - \mu_{p,1}^2\tau_1]. \tag{6a}$$

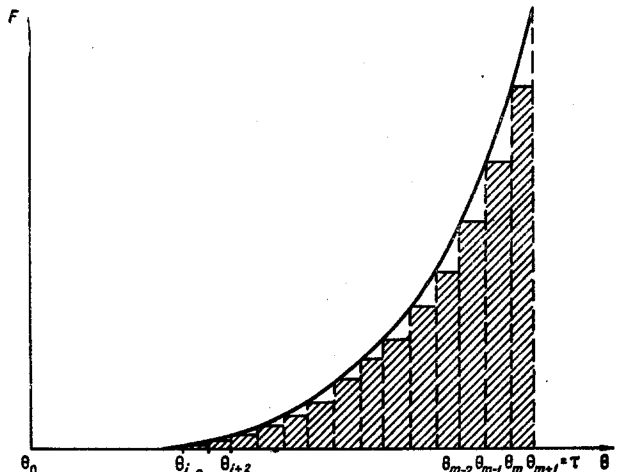
The solutions for subsequent time intervals are obtained in the same way as for the second interval. The expression for the temperature field at $\tau_m < \tau < \tau_{m+1}$ has the form

$$t_i(x,\ \tau) = \sum_{k=0}^{\infty} t_c^{(k)}(\tau) Q_{k,i,m+1}(x) + \sum_{j=1}^{m}\sum_{l=1}^{\infty}(-1)^l\sum_{k=l}^{\infty}\frac{t_c^{(k)}(0)}{(k-l)!}\tau_j^{k-l} \times \sum_{p=1}^{\infty}\left[\frac{\Phi_i(\mu_{p,j+1},\ x)}{\mu_{p,j+1}^{2l+1}\varphi'(\mu_{p,j+1})} - \frac{\Phi_i(\mu_{p,j},\ x)}{\mu_{p,j}^{2l+1}\varphi'(\mu_{p,j})}\right] \times \exp[-\mu_{p,m+1}^2(\tau-\tau_m) - \mu_{p,m}^2(\tau_m-\tau_{m-1}) - \cdots - \mu_{p,j+1}^2(\tau_{j+1}-\tau_j)] +$$

$$+\sum_{k=0}^{\infty}(-1)^k t_c^{(k)}(0)\sum_{p=1}^{\infty}\frac{\Phi_i(\mu_{o,1},\ x)}{\mu_{p,1}^{2k+1}\varphi'(\mu_{p,1})}\times$$

$$\times\exp[-\mu_{p,m+1}^2(\tau-\tau_m)-\mu_{p,m}^2(\tau_m-\tau_{m-1})-$$

$$-\ldots-\mu_{p,j+1}^2(\tau_{j+1}-\tau_j)-\ldots-\mu_{p,1}^2\tau_1]. \qquad (7)$$




Time dependence of the function F.

An analysis of (7) shows that the general expression for the temperature field can be divided into two parts, one of which,

$$\sum_{k=0}^{\infty} t_c^{(k)}(\tau)Q_{k,\,i}(x)\,, \qquad (8)$$

contains the distribution of the physical parameters with respect to the coordinate only at the instant of time considered ("running distribution"), while the other, which has m terms of the type (for fixed k, $l$)

$$\frac{t_c^{(k)}(0)}{(k-l)!}\ \tau_j^{k-l}\sum_{p=1}^{\infty}\left[\frac{\Phi_i(\mu_{p,\,j+1},\ x)}{\mu_{p,\,j+1}^{2l+1}\varphi'(\mu_{p,j+1})}-\right.$$

$$\left.-\frac{\Phi_i(\mu_{p,\,j},\ x)}{\mu_{p,\,j}^{2l+1}\varphi'(\mu_{p,\,j})}\right]\times$$

$$\times\exp[-\mu_{p,\,m+1}^2(\tau-\tau_m)-\mu_{p,\,m}^2(\tau_m-\tau_{m-1})-$$

$$-\ldots-\mu_{p,\,j+1}^2(\tau_{j+1}-\tau_j)], \qquad (8a)$$

contains the distributions of the parameters with respect to the coordinate for the entire process from the beginning to the instant considered.

We will consider how the temperature field is affected by the reduction of $\Delta\tau$ and $\Delta x$ with reference to the example of a thin plate, where it is possible to neglect the variation of the physical parameters with thickness. As $\Delta\tau_j$ tends to zero, the temperature distributions with respect to the coordinate at times $\tau_j$, $\tau_{j+1}$ will be indistinguishable, as a result of which the thermophysical parameters in the polynomials $Q_{k,\,i}(x)$ and temperature field (8) will be uniquely determined.

The absolute value of the sum of m terms of type (8a) can be represented graphically by the shaded area in the figure, where along the axis of abscissas we have plotted time, denoted by $\Theta$ (to distinguish it from the running time $\tau$, which is fixed in analyzing (8a)), and along the ordinate axis the quantity $F_{k,\,l}$:

$$F_{k,\,l}=\left\{\frac{t_c^{(k)}(0)}{(k-l)!}\ \Theta_j^{k-l}\left[\left(\frac{R^2}{a_{j+1}}\right)^l-\right.\right.$$

$$\left.-\left(\frac{R^2}{a_j}\right)^l\right]\times\sum_{p=1}^{\infty}A_{p,\,l}\times$$

$$\times\exp\left[-\nu_p^2\frac{a_{m+1}(\tau-\Theta_m)+\ldots+a_{j+1}(\Theta_{j+1}-\Theta_j)}{R^2}\right]\Bigg\}\times$$

$$\times\ [\Theta_{j+1}-\Theta_j]^{-1};$$

$$A_{p,\,l}=\frac{2(-1)^p}{\nu_p^{2l+1}}\cos\nu_p\frac{R-x}{R};$$

$$\nu_p=(2p-1)\ \frac{\pi}{2}. \qquad (9)$$

The expression in braces in (9) represents terms of type (8a) for a thin plate.

Treating the thermal diffusivity as a complex function of time, we find that as $\Delta\Theta\to 0$ and, correspondingly, as $m=\tau/\Delta\Theta\to\infty$, $F_{k,\,l}$ tends to the limit

$$F_{k,\,l}=\frac{t_c^{(k)}(0)\,l}{(k-l)!}\ \Theta^{k-l}\left[\frac{R^2}{a\,[t\,(\Theta)]}\right]^l\times$$

$$\times\ \frac{a'\,[t\,(\Theta)]}{a\,[t\,(\Theta)]}\ \frac{\partial t}{\partial\Theta}\sum_{p=1}^{\infty}A_{p,\,l}\exp\left[-\nu_p^2\frac{\int_{\Theta}^{\tau}a(\eta)\,d\eta}{R^2}\right]. \qquad (9a)$$

In this case the broken line on the graph (see figure) is transformed into a smooth curve, and the absolute value of the sum of m terms (8a) is expressed by the integral

$$\int_0^{\tau}F_{k,\,l}\,d\Theta, \qquad (10)$$

i.e., by the area under the curve in the figure.

Thus, the solution of the symmetrical problem for a thin plate with account for the variable physical parameters can be written in the form

$$t(x,\ \tau)=\sum_{k=0}^{\infty}t_c^{(k)}(\tau)\,Q_k(x)+$$

$$+\int_0^{\tau}\sum_{l=1}^{\infty}(-1)^l\sum_{k=l}^{\infty}F_{k,\,l}\,d\Theta+$$

$$+\sum_{k=0}^{\infty}(-1)^k t_c^{(k)}(0)\left[\frac{R^2}{a(t_{\text{in}})}\right]^k\times$$

$$\times\sum_{p=1}^{\infty}A_{p,\,k}\exp\left[-\nu_p^2\frac{\int_0^{\tau}a(\eta)\,d\eta}{R^2}\right]. \qquad (11)$$

An analysis shows that the effect of the variable physical parameters is felt only on a small time interval close to the calculation point $\Theta=\tau$; at greater distances from that point the effect of a change in the parameters rapidly diminishes—as $\Theta$ decreases, the value of $F_{k,\,l}$ asymptotically approaches the axis of abscissas (see figure). Therefore, in analyzing the temperature fields it is possible to use a small time interval $[\Theta, \tau]$.

When $a'(t) \to 0$ and, consequently, $a(t) \to a$, the integral in (11) tends to zero and (11) coincides with the general expression for the temperature field in the linear problem [4].

If the temperature at the boundary is given as a function $t_c(\alpha\tau)$, then as $\alpha \to 0$ all the terms in (11) will be infinitesimals, except $t_c(\alpha\tau)Q_0$, where $Q_0 = 1$. Consequently, the temperature does not vary over the thickness of the plate and follows the variation of the temperature at the boundary. This corresponds to normal physical notions; if the temperature at the boundary varies slowly ($\alpha \to 0$), the temperature field in the plate will be able to equalize itself.

The basic laws obtained from an investigation of the symmetrical temperature field in a thin plate are also preserved in the general case; however, their analytic expression is much more complicated. Treating the eigennumbers $\mu_p$ and the functions $\Phi$ and $\varphi$, which contain values of the temperature-dependent thermophysical parameters, as complex functions of $t(\Theta)$, we represent the solution of the general problem in form (11), where the following expression can be given for $F_{k,l}$:

$$F_{k,l} = \frac{t_c^k(0)}{(k-l)!}\Theta^{k-l}\sum_{p=1}^{\infty}\frac{d}{d\Theta} \times \left[\frac{\Phi_l[\Theta, \mu_p(\Theta), x]}{\mu_p^{2l+1}(\Theta)\varphi'[\Theta, \mu_p(\Theta)]}\right] \times \exp\left[-\int_{\Theta}^{\tau}\mu_p^2(\eta)\,d\eta\right]. \tag{12}$$

It is more convenient to use the general expression (7), (12) not for specific calculations (since in the general case $F_{k,l}$ cannot be expressed in explicit form), but for analyzing the laws of thermal processes in multilayer systems: the effect of the temperature dependence of the parameters, the law of variation of the ambient temperature, the relative distribution of the layers of the multilayer system, etc.

The general solution can also be used for simplified calculations, when the temperature field is determined from the part of the solution characterized by a running distribution of the physical parameters with respect to the coordinate; the other part, which takes into account the effect of the previous variation of the thermal properties, can serve to estimate the error of the approximate calculations.

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